4.2 Limit of a Vector Function

Definition. A vector function $\vec{f}(t)$ is said to tend to a vector limit $\vec{l}$ as $t \to a$, if corresponding to any positive number ε that we may choose, no matter how small, there exists a positive number δ such that $|\vec{f}(t) - \vec{l}| < \varepsilon$ for $|t-a| \leqslant \delta$.

This is expressed by writing

$$\lim_{t \to a} \vec{f}(t) = \vec{l}.$$

We also say that $\vec{f}(t)$ tends to $\vec{l}$ as t tends to a; and express this by $\vec{f}(t) \to \vec{l}$ as $t \to a$.

4.3 Fundamental Theorems on Limits

If $\vec{f}(t)$, $\vec{g}(t)$ be vector functions and $\varphi(t)$ a scalar function of t such that as $t \to a$

$$\vec{f}(t) \to \vec{l},\ \vec{g}(t) \to \vec{m} \text{ and } \varphi(t) \to n,$$

then (i) $\vec{f}(t) + \vec{g}(t) \to \vec{l} + \vec{m}$, as $t \to a$,

(ii) $\vec{f}(t) - \vec{g}(t) \to \vec{l} - \vec{m}$, as $t \to a$,

(iii) $\varphi(t)\vec{f}(t) \to n\vec{l}$, as $t \to a$,

(iv) $\vec{f}(t) . \vec{g}(t) \to \vec{l} . \vec{m}$, as $t \to a$,

(v) $\vec{f}(t) \times \vec{g}(t) \to \vec{l} \times \vec{m}$, as $t \to a$.

Proof. Choose a positive number ϵ, no matter how small.

We have $\lim_{t \to a} \vec{f}(t) = \vec{l}$ and $\lim_{t \to a} \vec{g}(t) = \vec{m}$.

By the definition of limit we can choose δ_1 and δ_2 so that

$$|\vec{f}(t)-\vec{l}| < \tfrac{1}{2}\varepsilon \text{ for } |t-a| \leqslant \delta_1$$

and
$$|\vec{g}(t)-\vec{m}| < \tfrac{1}{2}\varepsilon \text{ for } |t-a| \leqslant \delta_2.$$

If we denote the smaller of δ_1, δ_2 by δ, then each of the above two inequalities will hold for $|t-a| \leqslant \delta$,

that is, $|\vec{f}(t)-\vec{l}| < \tfrac{1}{2}\varepsilon$ and $|\vec{g}(t)-\vec{m}| < \tfrac{1}{2}\varepsilon$ for $|t-a| \leqslant \delta$.

(i) We have $|\vec{f}(t)+\vec{g}(t)-(\vec{l}+\vec{m})| \leqslant |\vec{f}(t)-\vec{l}|$

$$+ |\vec{g}(t)-\vec{m}|.$$

Hence, when $|t-a| \leqslant \delta$,

$$|\vec{f}(t)+\vec{g}(t)-(\vec{l}+\vec{m})| < \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon.$$

Hence, by the definition of limit,

$$\vec{f}(t)+\vec{g}(t)\to\vec{l}+\vec{m}, \text{ as } t\to a.$$

We leave the proofs of the remaining theorems as an exercise for the reader.

4.4 Continuity of a Vector Function at a Point

Definition. The vector function $\vec{f}(t)$, defined over the interval I, is said to be continuous at $t=a\epsilon I$ if $\lim_{t\to a} \vec{f}(t)=\vec{f}(a)$.

This definition can be put in the distance form, that is, modulus form, as follows—

A vector function $\vec{f}(t)$, defined over the interval I, is said to be continuous at $t=a\varepsilon I$, if given $\varepsilon>0$ there exists a positive number $\delta(a, \varepsilon)$ such that $|\vec{f}(t)-\vec{f}(a)| < \varepsilon$ for $|t-a| < \delta$.

A vector function, $\vec{f}(t)$ which is not continuous at $t=a$, is called discontinuous at $t=a$.

4.5 Continuity of a Vector Function in an interval

A vector function $\vec{f}(t)$ defined over the interval I is said to be continuous in the interval $[a, b]\subseteq I$ if $\vec{f}(t)$ is continuous for all values of t such that $a\leqslant t\leqslant b$.

4.6 Properties of Continuous Vector Functions

Theorem. If $\vec{f}(t)$ and $\vec{\varphi}(t)$ are continuous at $t=a$, then

(i) $\vec{f}(t)+\vec{\varphi}(t)$ is continuous at $t=a$;

(ii) $\vec{f}(t)-\vec{\varphi}(t)$ is continuous at $t=a$;

(iii) $\vec{f}(t)\psi(t)$ is continuous at $t=a$, where $\psi(t)$, a scalar function of t, is continuous at $t=a$;

(iv) $\dfrac{\vec{f}(t)}{\psi(t)}$ is continuous at $t=a$, where $\psi(t)$, a scalar function of t, is continuous at $t=a$ and $\psi(t)\neq 0$ for any value of t in the domain of definition;

(v) $\vec{f}(t).\vec{\varphi}(t)$ is continuous at $t=a$;

(vi) $\vec{f}(t)\times\vec{\varphi}(t)$ is continuous at $t=a$.

Proof. As $\vec{f}(t)$ and $\vec{\varphi}(t)$ are continuous Rt $t=a$, therefore, by the definition of continuity,

$$\lim_{t\to a}\vec{f}(t)=\vec{f}(a) \text{ and } \lim_{t\to a}\vec{\varphi}(t)=\vec{\varphi}(a).$$

(i) Let $\vec{F}(t)=\vec{f}(t)+\vec{\varphi}(t)$.

Then $\vec{F}(a)=\vec{f}(a)+\vec{\varphi}(a)$.

Now $\lim_{t\to a}\vec{F}(t)=\lim_{t\to a}[\vec{f}(t)+\vec{\varphi}(t)]$

$$=\lim_{t\to a}\vec{f}(t)+\lim_{t\to a}\vec{\varphi}(t),$$

by the theorem on limit,

$$=\vec{f}(a)+\vec{\varphi}(a)=\vec{F}(a).$$

Hence, by the definition of continuity,

$$\vec{F}(t), \text{ that is, } \vec{f}(t)+\vec{\varphi}(t)$$

is continuous at $t=a$.

We leave the proofs of the other theorems as an exercise for the reader.

4.7 Differentiability of a Vector Function at a point

(R. U. 1986)

Definition. A vector function $\vec{f}(t)$ defined in the interval I is said to be differentiable at $t = a \in I$ if

$$\lim_{t \to a} \frac{\vec{f}(t) - \vec{f}(a)}{t - a} \text{ exists, in other words if}$$

$$\lim_{\delta a \to 0} \frac{\vec{f}(a + \delta a) - \vec{f}(a)}{\delta a} \text{ exists.}$$

The value of the limit is called the **derivative** (differential co-efficient) of $\vec{f}(t)$ at $t = a$ and it is denoted by

$$\frac{d\vec{f}}{dt} \quad \text{or} \quad \vec{f}'(t).$$

Clearly this is a vector quantity.

Thus
$$\boxed{\frac{d\vec{f}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t}.}$$

Note. Obviously, by definition $\boxed{\frac{d\vec{a}}{dt} = \vec{0}}$ if $\vec{a}$ is a constant vector.

4.8 Theorem. If a vector function is differentiable finitely at a point, then it must be continuous at that point.

Proof. Let the vector function $\vec{f}(t)$ be differentiable finitely at $t = a$.

Then, by the definition of differentiability of a vector function at a point,

$$\lim_{h \to 0} \frac{\vec{f}(a + h) - \vec{f}(a)}{h} = \vec{f}'(a), \text{ which is finite.}$$

Now $\lim\limits_{t \to a} \vec{f}(t) = \lim\limits_{h \to 0} \vec{f}(a + h)$

$$= \lim_{h \to 0} [\vec{f}(a + h) - \vec{f}(a) + \vec{f}(a)]$$

$$= \lim_{h\to 0} \left[\frac{\vec{f}(a+h)-\vec{f}(a)}{h}.h+\vec{f}(a) \right]$$

$$= \lim_{h\to 0} \frac{\vec{f}(a+h)-\vec{f}(a)}{h}. \lim_{h\to 0} h+\vec{f}(a)$$

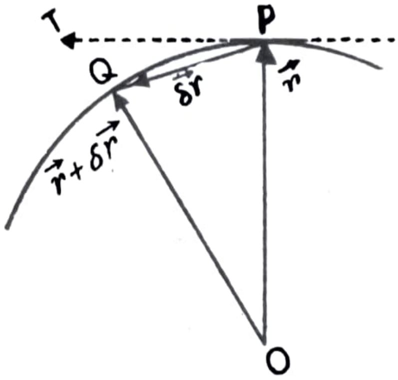
$$= \vec{f}'(a).0+\vec{f}(a)=\vec{f}(a).$$

Hence $\vec{f}(t)$ is continuous at $t=a$.

Note. The converse of this theorem is not necessarily true.

4.9 Geometrical interpretation of the Derivative

Let us consider a point P on a curve $\vec{r}=\vec{f}(t)$, with position vector $\vec{r}$ given by a parameter t. Intuitively we see that a change in the value of t causes a change in the value of $\vec{r}$. Let Q be the point $\vec{r}+\delta\vec{r}$ on the curve when t changes to $t+\delta t$.



Then $\vec{OP}=\vec{r}=\vec{f}(t)$

and $\vec{OQ}=\vec{r}+\delta\vec{r}$

$=\vec{f}(t+\delta t)$.

$\therefore \quad \delta\vec{r}=(\vec{r}+\delta\vec{r})-\vec{r}$

$=\vec{f}(t+\delta t)-\vec{f}(t)=\vec{PQ}$

or $\quad \dfrac{\delta\vec{r}}{\delta t}=\dfrac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$

$=\dfrac{\vec{PQ}}{\delta t}$.

Now in the limit as $\delta t\to 0$,

$Q\to P$ and $\vec{PQ}\to$ the tangent $\vec{PT}$ at P.

Hence $\dfrac{d\vec{r}}{dt}$ is a vector parallel to the tangent at P in the sense of t increasing.

Note. (i) If s be the arc length then $|\delta\vec{r}|\to\delta s$ and so $\dfrac{d\vec{r}}{ds}$ becomes the unit tangent to the curve $\vec{r}=\vec{f}(s)$.

(ii) $\frac{d\vec{r}}{dt}$ or $\vec{f}'(t)$ is called the first derivative of $\vec{r}$ provided $\vec{r} = \vec{f}(t)$ is a derivable function.

If the first derivative is also derivable then its derivative is called the second derivative of $\vec{r}$ and is denoted by

$$\frac{d^2\vec{r}}{dt^2} \text{ or } \vec{f}''(t).$$

Similarly the third derivative is $\frac{d^3\vec{r}}{dt^3}$ or $\vec{f}'''(t)$ and so on.

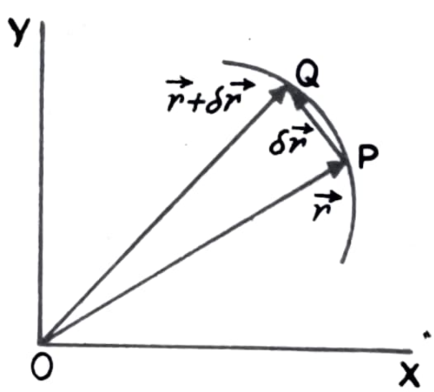
In general the nth derivative is $\frac{d^n\vec{r}}{dt^n}$ or $\vec{f}^n(t)$.

4.10 Physical interpretation of the first and the second Derivatives.

Let the vector equation of the curve be $\vec{r} = \vec{f}(t)$.

Let us consider a point P on the curve, with position vector $\vec{r}$ given by a parameter t.

Intuitively we see that a change in the value of t causes a change in the value of $\vec{r}$. Let Q be the point $\vec{r} + \delta\vec{r}$ on the curve when t changes to $t+\delta t$.



Then $\delta\vec{r}$ is the displacement of the point P in time δt, when the parameter t denotes the time. $\frac{\delta\vec{r}}{\delta t}$ is called the average rate of change of $\vec{r}$ with t, that is, the average velocity during the interval.

And $\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \text{velocity } \vec{v}$.

Thus $\frac{d\vec{r}}{dt}$ is the velocity $\vec{v}$ at P which is in the direction of the tangent of the path of the point.

Similarly, the acceleration $\vec{a}$ at P is the rate of change of the velocity $\vec{v}$ and is given by

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta \vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}.$$

4.11 Differentiation of a Vector Function of a Scalar Function

Let $\vec{f}(t)$ be a vector function of the scalar variable t and t be a function of another scalar s.

Let δt be a small increment in t and $\delta \vec{f}$ and δs be the corresponding small increments in $\vec{f}$ and s respectively.

Then, when $\delta t \to 0$, $\delta \vec{f} \to 0$ and $\delta s \to 0$.

Now $$\frac{\delta \vec{f}}{\delta s} = \frac{\delta \vec{f}}{\delta t} \cdot \frac{\delta t}{\delta s}.$$

Taking the limits as $\delta t \to 0$ and consequently $\delta s \to 0$, we have

$$\lim_{\delta s \to 0} \frac{\delta \vec{f}}{\delta s} = \lim_{\delta t \to 0} \frac{\delta \vec{f}}{\delta t} \cdot \lim_{\delta s \to 0} \frac{\delta t}{\delta s}$$

or $$\frac{d\vec{f}}{ds} = \frac{d\vec{f}}{dt} \cdot \frac{dt}{ds}.$$